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The 15th Summer course for Behavior Modeling in Transportation Networks
@The University of Tokyo

Advanced behavior models

Recent development of discrete choice models

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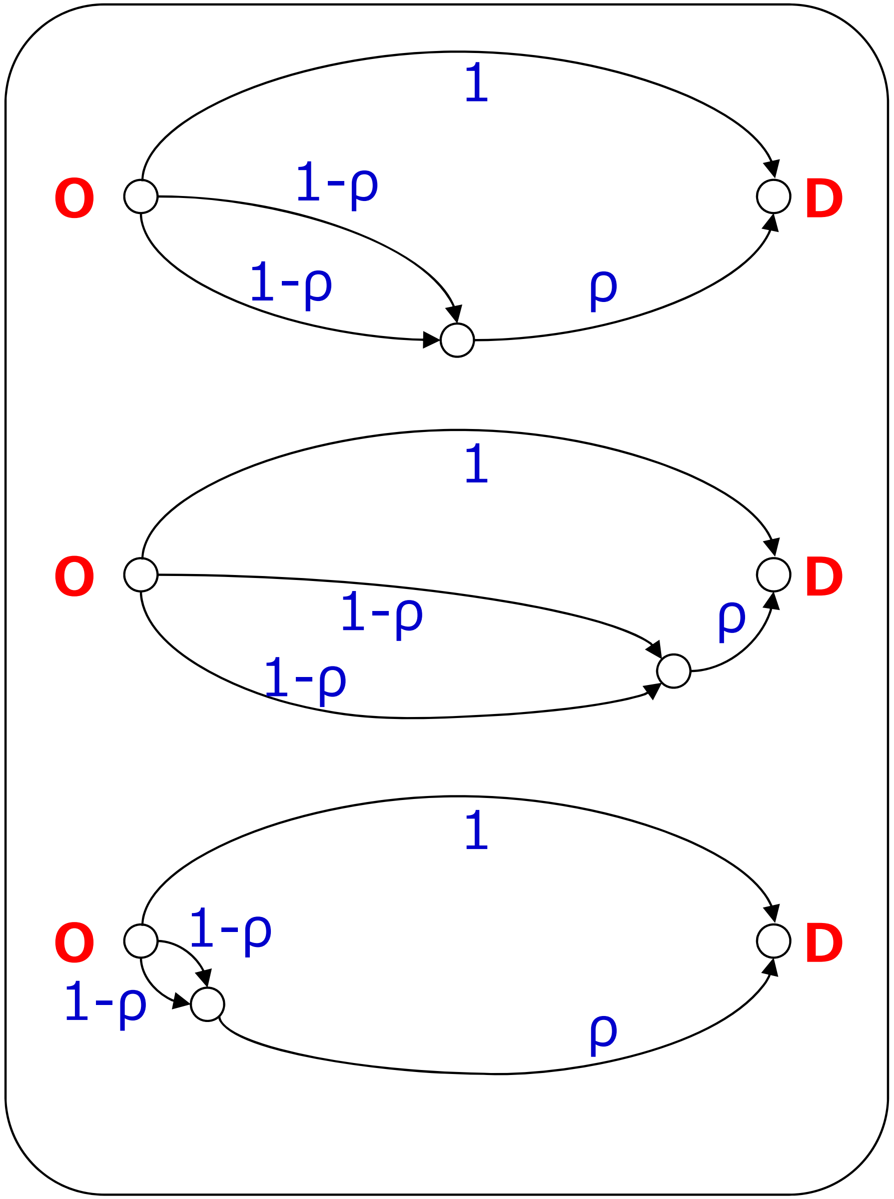
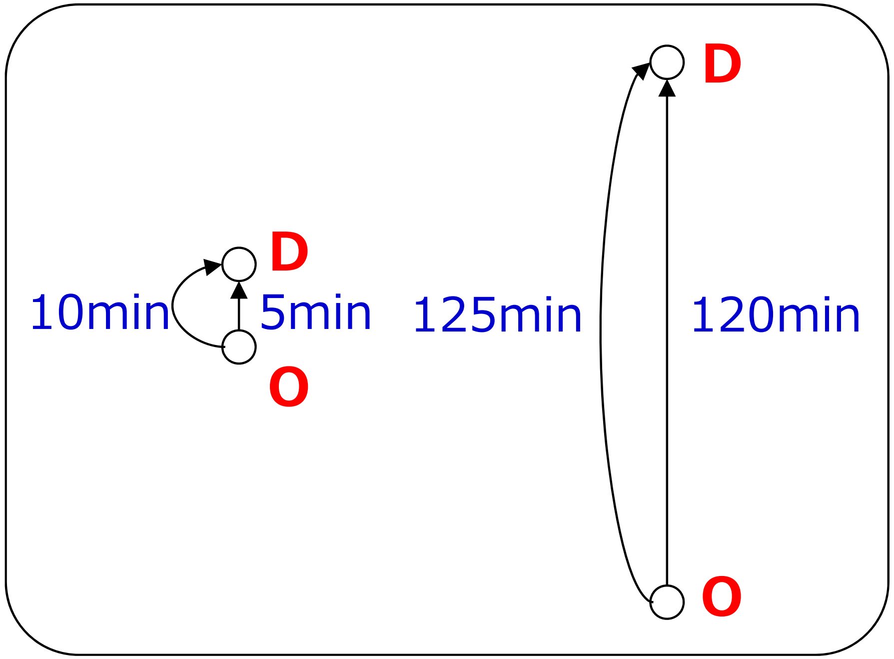
Why advanced models are needed? A case of route choice

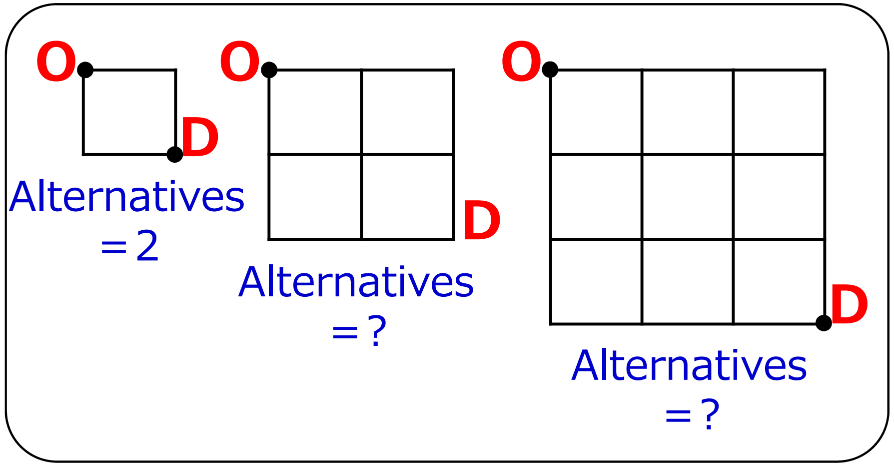
(a)Route overlap



(b) Route length



(c) Route enumeration



Closed-form and open-form

- **Closed-form expression**
  - A mathematical expression that can be evaluated in a finite number of operations

example

$$P_{ij} = \frac{e^{\beta x_{ij}/\lambda_k} \left(\sum_{j \in B_k} e^{\beta x_{ij}/\lambda_l} \right)^{\lambda_k - 1}}{\sum_{l=1}^{K} \left(\sum_{j \in B_k} e^{\beta x_{ij}/\lambda_l} \right)^{\lambda_l}}$$

- **Open-form expression**

example

$$P_{ij} = \int_{\beta_i \in D_{\beta_i}} \frac{\exp\left(\beta_i x_{ij}\right)}{\sum_{j'=1}^{J} \exp\left(\beta_i x_{ij}\right)} f(\beta_i) d\beta_i$$

Pros and cons

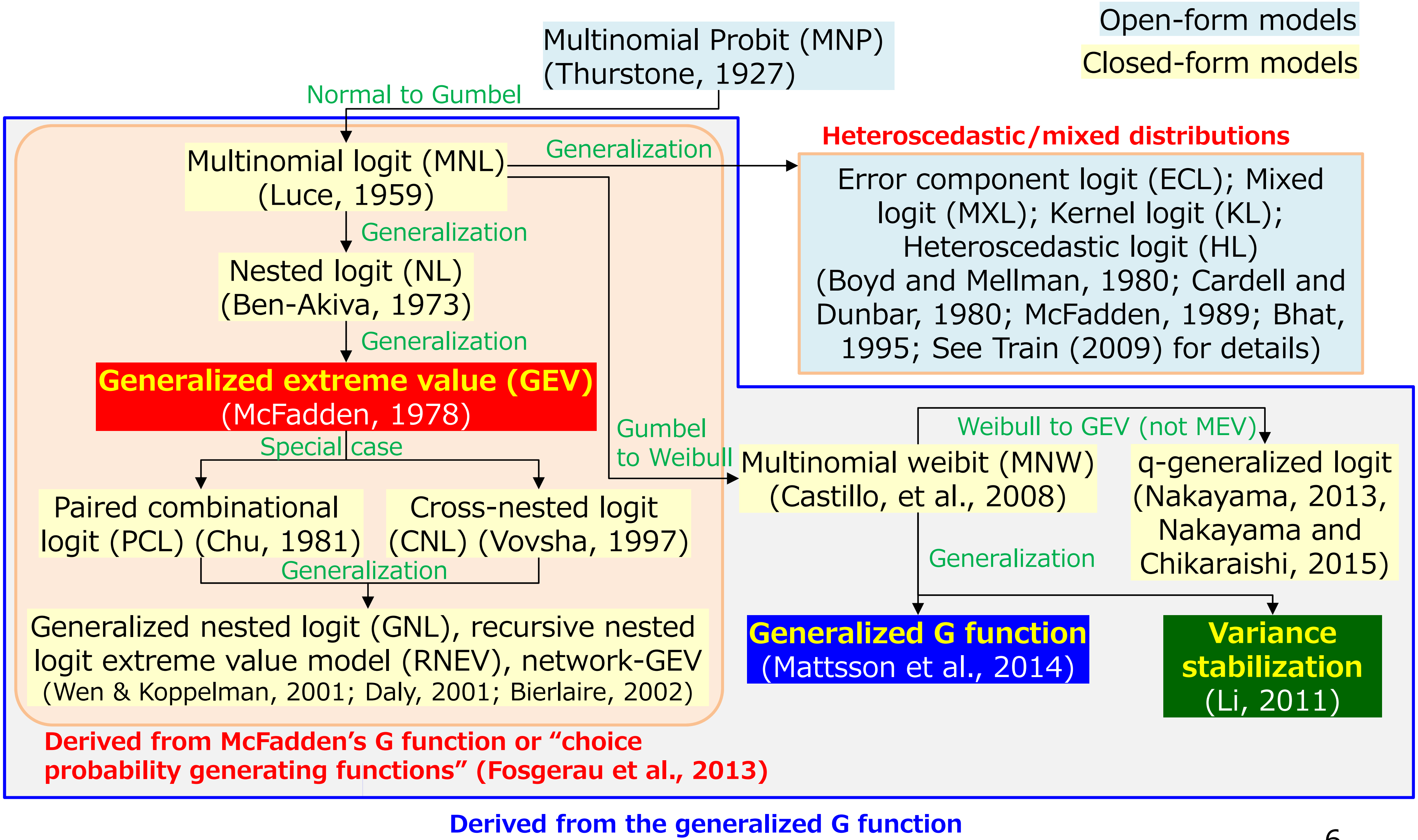
- **Closed-form expression**
  - Pros
    - Easy to use in practice
    - Can be embedded into a larger modeling system as a subcomponent
  - Cons
    - Not flexible enough in some cases
- **Open-form expression**
  - Pros
    - Very flexible and any kind of closed-form models can be approximately modeled
  - Cons
    - Behavioral understanding of the model is sometimes difficult

Contents (closed-form models)

1. **McFadden's G function** (McFadden, 1978)
   - Route overlap
2. **Generalized G function** (Mattsson et al., 2014)
   - Route overlap and route length
3. **Recursive logit** (Fosgerau et al., 2013)
   - Route enumeration

Discrete choice models

[based on Hato (2002)]



(McFadden, 1978)

McFadden's G function

The properties that the G function must exhibit

① $G(y_{i1}, y_{i2}, \dots, y_{iJ_i}) \geq 0$

② G is homogeneous of degree m : $G(\alpha y_{i1}, \dots, \alpha y_{iJ_i}) = \alpha^m G(y_{i1}, \dots, y_{iJ_i})$

③ $\lim_{y_{ij} \to \infty} G(y_{i1}, y_{i2}, \dots, y_{iJ_i}) = \infty$ for any j

④ The cross partial derivatives of G satisfy:

$$(-1)^{k-1} \cdot \frac{\partial^k G(y_{i1}, y_{i2}, \dots, y_{iJ_i})}{\partial y_{i1} \partial y_{i2} \cdots \partial y_{ik}} \geq 0$$

When all conditions are satisfied, the choice probability can be defined as:

$$P_{ij} = \frac{e^{V_{ij}} \cdot G_j(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ_i}})}{G(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ_i}})}$$

(where, $G_j = \partial G / \partial y_{ij}$)

Assumption: $F(\epsilon_{i1}, \dots, \epsilon_{iJ}) = \exp\{-G(e^{-\epsilon_{i1}}, \dots, e^{-\epsilon_{iJ}})\}$ ※ $u_{ij} = V_{ij} + \epsilon_{ij}$

Derivation of choice probability

Suppose $u_{ij} = V_{ij} + \epsilon_{ij}$, where $(\epsilon_{i1}, \dots, \epsilon_{iJ})$ is distributed F defined as:

$$F(\epsilon_{i1}, \dots, \epsilon_{iJ}) = \exp\{-G(e^{-\epsilon_{i1}}, \dots, e^{-\epsilon_{iJ}})\}$$

multivariate extreme value (MEV) distribution (**NOT** GEV)

Then, the probability of the first alternative P_{i1} satisfies:

$$P_{i1} = \int_{\epsilon=-\infty}^{+\infty} F_1(\epsilon, V_{i1} - V_{i2} + \epsilon, \dots, V_{i1} - V_{iJ} + \epsilon) d\epsilon$$

$$= \int_{\epsilon=-\infty}^{+\infty} \begin{bmatrix} e^{-\epsilon} G_1(e^{-\epsilon}, e^{-\epsilon - V_{i1} + V_{i2}}, \dots, e^{-\epsilon - V_{i1} + V_{iJ}}) \\ \times \exp\{-G(e^{-\epsilon}, e^{-\epsilon - V_{i1} + V_{i2}}, \dots, e^{-\epsilon - V_{i1} + V_{iJ}})\} \end{bmatrix} d\epsilon$$

$$= \int_{\epsilon=-\infty}^{+\infty} \begin{bmatrix} e^{-\epsilon} G_1(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ}}) \\ \times \exp\{-e^{-\epsilon} e^{-V_{i1}} G(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ}})\} \end{bmatrix} d\epsilon$$

Uses the linear homogeneity

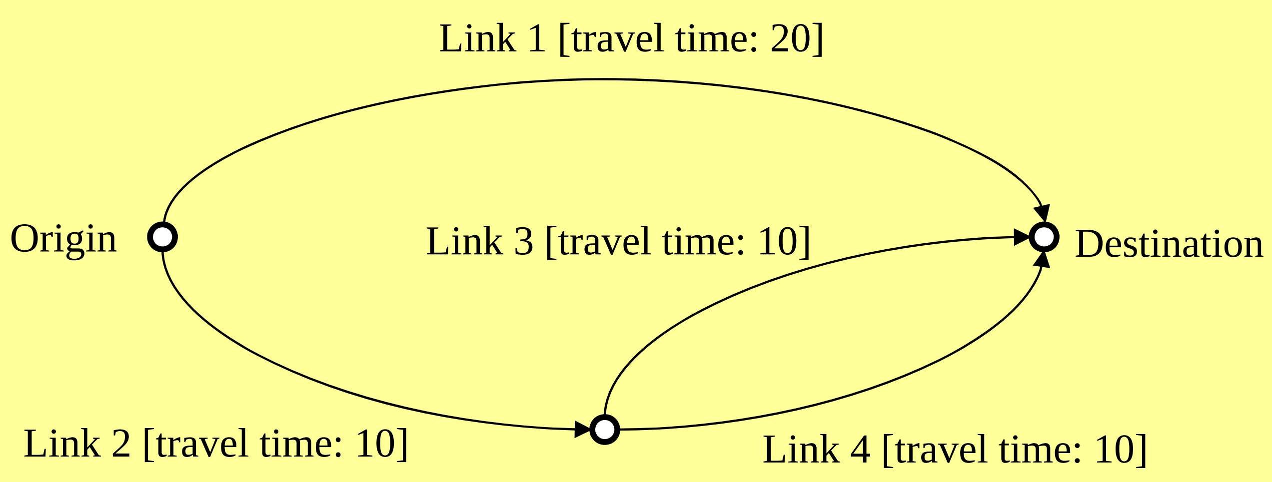
$$= \frac{e^{V_{i1}} G_1(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ}})}{G(e^{V_{i1}}, e^{V_{i2}}, \dots, e^{V_{iJ}})}$$

Some examples

	G function	Choice probability
Logit	$G = \Sigma_{j=1}^{J} y_{ij}$	$P_{ij} = \dfrac{\exp(V_{ij})}{\Sigma_{j'=1}^{J} \exp(V_{ij'})}$
Nested logit	$G = \Sigma_{l=1}^{K} \left(\Sigma_{j \in B_l} y_{ij}^{1/\lambda_l}\right)^{\lambda_l}$	$P_{ij} = \dfrac{e^{V_{ij}/\lambda_k}\left(\Sigma_{j \in B_k} e^{V_{ij}/\lambda_l}\right)^{\lambda_k - 1}}{\Sigma_{l=1}^{K}\left(\Sigma_{j \in B_k} e^{V_{ij}/\lambda_l}\right)^{\lambda_l}}$
Paired combinational logit	$G = \Sigma_{k=1}^{J-1} \Sigma_{l=k+1}^{J} \left(y_{ik}^{1/\lambda_{kl}} + y_{il}^{1/\lambda_{kl}}\right)^{\lambda_{kl}}$	$P_{ij} = \dfrac{\sum_{m \neq j} e^{\frac{V_{ij}}{\lambda_{jm}}}\left(e^{\frac{V_{ij}}{\lambda_{jm}}} + e^{\frac{V_{im}}{\lambda_{jm}}}\right)^{\lambda_{jm} - 1}}{\Sigma_{k=1}^{J-1} \Sigma_{l=k+1}^{J} \left(e^{\frac{V_{ik}}{\lambda_{kl}}} + e^{\frac{V_{il}}{\lambda_{kl}}}\right)^{\lambda_{kl}}}$
Generalized nested logit	$G = \Sigma_{k=1}^{K} \left(\Sigma_{j \in B_k} (\alpha_{jk} y_{ij})^{1/\lambda_k}\right)^{\lambda_k}$	$P_{ij} = \dfrac{\Sigma_k \left(\alpha_{jk} e^{V_{ij}}\right)^{\frac{1}{\lambda_k}} \left(\Sigma_{m \in B_k} \left(\alpha_{mk} e^{V_{im}}\right)^{\frac{1}{\lambda_k}}\right)^{\lambda_k - 1}}{\Sigma_{l=1}^{K} \left(\Sigma_{m \in B_k} \left(\alpha_{ml} e^{V_{im}}\right)^{\frac{1}{\lambda_l}}\right)^{\lambda_l}}$

$* \ y_{ij} \coloneqq \exp(V_{ij})$

Illustration



Path 1 = {Link 1} [travel time: 20]
Path 2 = {Link 2, Link 3} [travel time: 20]
Path 3 = {Link 2, Link 4} [travel time: 20]

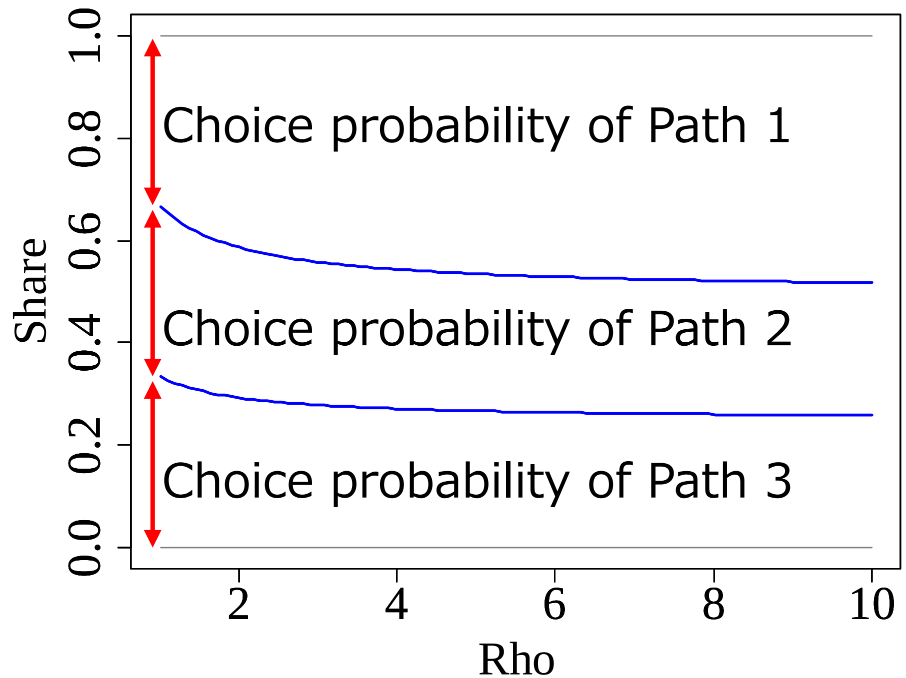
Nested logit

$$P_1 = \frac{\exp(\beta x)}{\exp(\beta x) + \exp(\frac{1}{\rho}\Lambda)}$$

$$P_2 = P_3 = \frac{1}{2} \cdot \frac{\exp(\frac{1}{\rho}\Lambda)}{\exp(\beta x) + \exp(\frac{1}{\rho}\Lambda)}$$

※ $\Lambda = \ln(\exp(\rho\beta x) + \exp(\rho\beta x))$

β is fixed as -0.2



(Mattsson et al., 2014)

Generalized G (A) function

The properties that the A function must exhibit

① $A(y_{i1}, y_{i2}, \dots, y_{iJ_i}) \geq 0$

② A is homogeneous of degree one: $A(\alpha y_{i1}, \dots, \alpha y_{iJ_i}) = \alpha A(y_{i1}, \dots, y_{iJ_i})$

③ $\lim_{y_{ij} \to \infty} A(y_{i1}, y_{i2}, \dots, y_{iJ_i}) = \infty$ for any j

④The cross partial derivatives of A satisfy:

$$(-1)^{k-1} \cdot \frac{\partial^k A(y_{i1}, y_{i2}, \dots, y_{iJ_i})}{\partial y_{i1} \partial y_{i2} \cdots \partial y_{ik}} \geq 0$$

When all conditions are satisfied, the choice probability can be defined as:

$$P_{ij} = \frac{w_{ij} \cdot A_j(w_{i1}, w_{i2}, \dots, w_{iJ})}{A(w_{i1}, w_{i2}, \dots, w_{iJ})}$$

(where, $A_j = \partial A / \partial w_{ij}$)

Assumption: $F(x_{i1}, \dots, x_{iJ}) = \exp\{-A(-w_{i1} \ln[\Psi(x_{i1})], \dots, -w_{iJ} \ln[\Psi(x_{iJ})])\}$

When $w_j = e^{V_{ij}}$ and $\Psi(x_j) \sim i.i.d.\,Gumbel$, A function becomes McFadden's G function

(Mattsson et al., 2014)

Derivation of choice probability

Note that $\Pr[\max_{j \in J} X_{ij} \leq x] = F(x, x, \dots, x)$, where F is defined as:

$$F(x_{i1}, \dots, x_{iJ}) = \exp\{-A\left(-w_{i1}\ln[\Psi(x_{i1})], \dots, -w_{iJ}\ln\left[\Psi\left(x_{iJ}\right)\right]\right)\}$$

Then, the probability of the first alternative P_{i1} satisfies:

$$P_{i1} = \int_{x \in \Omega_i} F_1(x, x, \dots, x)dx$$

$$= \int_{x \in \Omega_i} \left[\begin{array}{c} e^{-A\left(-w_{i1}\ln[\Psi(x)], \dots, -w_{iJ}\ln[\Psi(x)]\right)} \times \\ A_1\left(-w_{i1}\ln[\Psi(x)], \dots, -w_{iJ}\ln[\Psi(x)]\right) \cdot w_{i1} \cdot \dfrac{\psi(x)}{\Psi(x)} \end{array} \right] dx$$

$$= w_{i1} \cdot \frac{A_1(w)}{A(w)} \int_{x \in \Omega_i} \underbrace{A(w)[\Psi(x)]^{A(w)-1}\psi(x)dx}$$

=density function of F

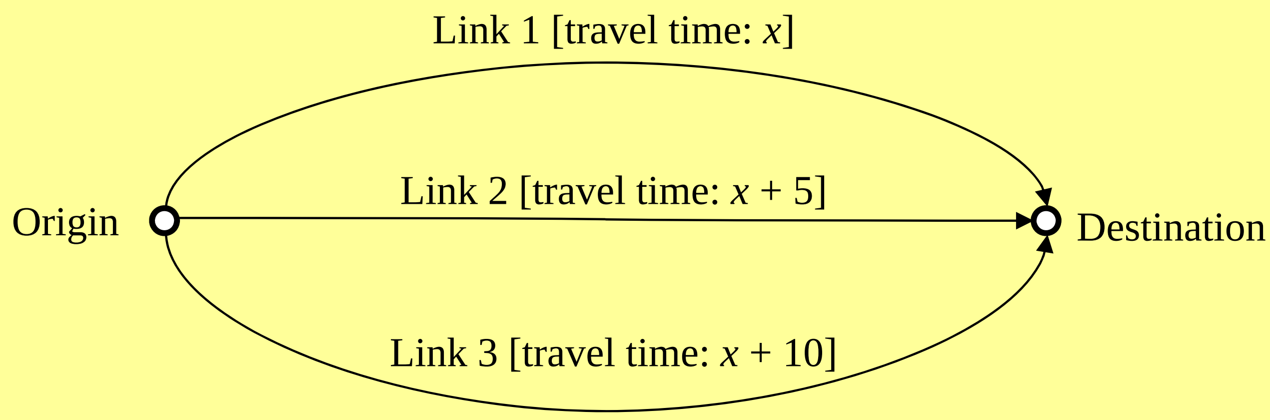
Uses the linear homogeneity

$$= w_{i1} \cdot \frac{A_1(w)}{A(w)}$$

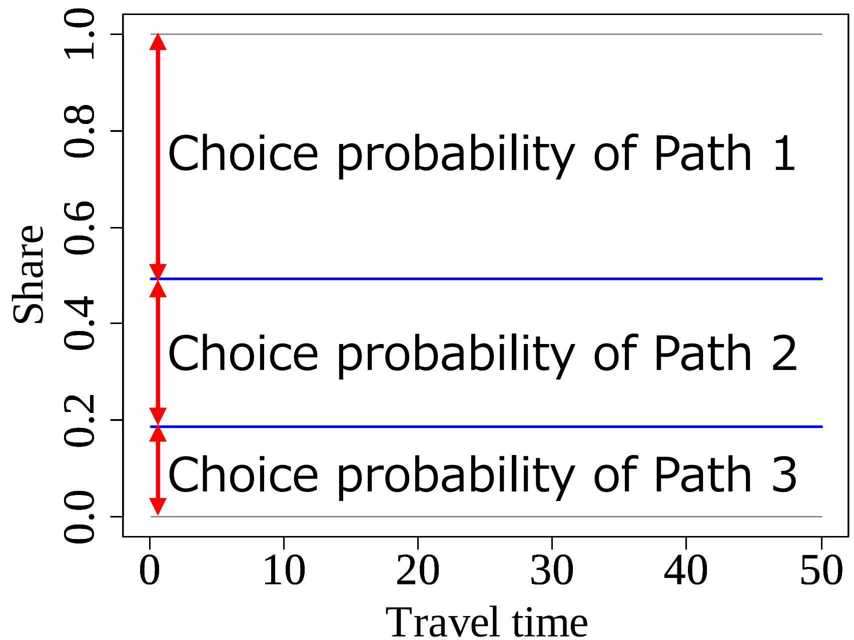
Some examples

	G function	Choice probability
Under the assumption of independence (Mattsson et al., 2014)		
Logit (Gumbel)	A: summation, $w_{ij} = e^{\beta V_{ij}}$, $\Psi(x_{ij}) \sim Gumbel(\beta, 0)$	$P_{ij} = \frac{\exp(\beta V_{ij})}{\Sigma_{j'=1}^{J} \exp(\beta V_{ij'})}$
Weibit-type (Frechet)	A: summation, $w_{ij} = {V_{ij}}^{\beta}$, $\Psi(x_{ij}) \sim Frechet(\beta, 1)$	$P_{ij} = \frac{V_{ij}^{\beta}}{\Sigma_{j'=1}^{J} V_{ij'}^{\beta}}$
Weibit (Weibull)	A: summation, $w_{ij} = {V_{ij}}^{-\beta}$, $\Psi(x_{ij}) \sim Weibull(\beta, 1)$	$P_{ij} = \frac{V_{ij}^{-\beta}}{\Sigma_{j'=1}^{J} V_{ij'}^{-\beta}}$
Under the statistical dependence (Chikaraishi and Nakayama, 2016)		
Nested logit	$A = \Sigma_{l=1}^{K} \left(\Sigma_{j \in B_l} w_{ij}^{1/\lambda_l} \right)^{\lambda_l}$, $w_{ij} = e^{\beta(a_{il} + b_{ij})}$, $\Psi(x_{ij}) \sim Gumbel(\beta, 0)$	$P_{ij} = \frac{\exp\left[\frac{\beta b_{ij}}{\lambda_l}\right]}{\Sigma_{j' \in J_l} \exp\left[\frac{\beta b_{ij'}}{\lambda_l}\right]} \cdot \frac{\exp[\beta a_{il} + \lambda_l \bar{b}_{0il}]}{\Sigma_{l'=1}^{L} \exp[\beta a_{il'} + \lambda_{l'} \bar{b}_{0il'}]}$ $\bar{b}_{0il} = \ln \sum_{j \in J_l} \exp(\beta b_{ij} / \lambda_l)$
Nested weibit	$A = \Sigma_{l=1}^{K} \left(\Sigma_{j \in B_l} w_{ij}^{1/\lambda_l} \right)^{\lambda_l}$, $w_{ij} = (a_{il} b_{ij})^{-\beta}$ $\Psi(x_{ij}) \sim Weibull(\beta, 1)$	$P_{ij} = \frac{b_{ij}^{-\frac{\beta}{\lambda_l}}}{\Sigma_{j' \in J_l} b_{ij'}^{-\frac{\beta}{\lambda_l}}} \cdot \frac{(a_{il})^{-\beta} \left(\bar{\bar{b}}_{0il}\right)^{\lambda_l}}{\Sigma_{l'=1}^{L} (a_{il'})^{-\beta} \left(\bar{\bar{b}}_{0il'}\right)^{\lambda_{l'}}}$ $\bar{\bar{b}}_{0il} = \sum_{j \in J_l} b_{ij}^{-\beta/\lambda_l}$

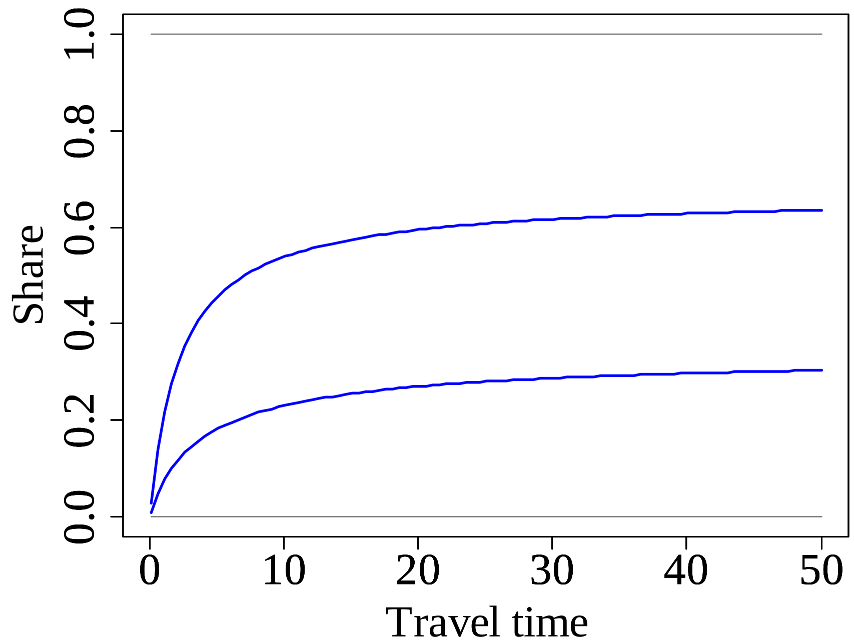
Illustration



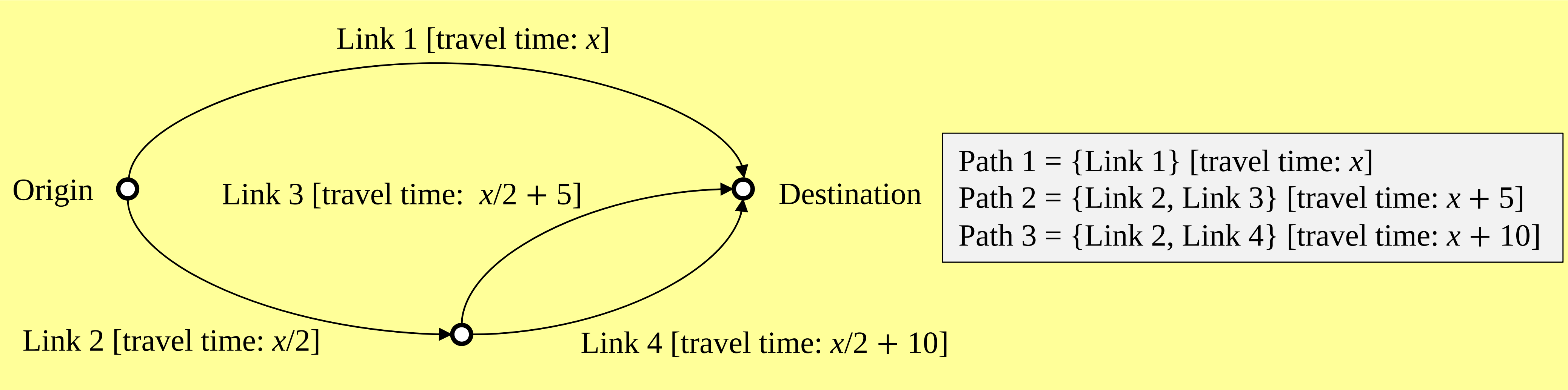
Logit



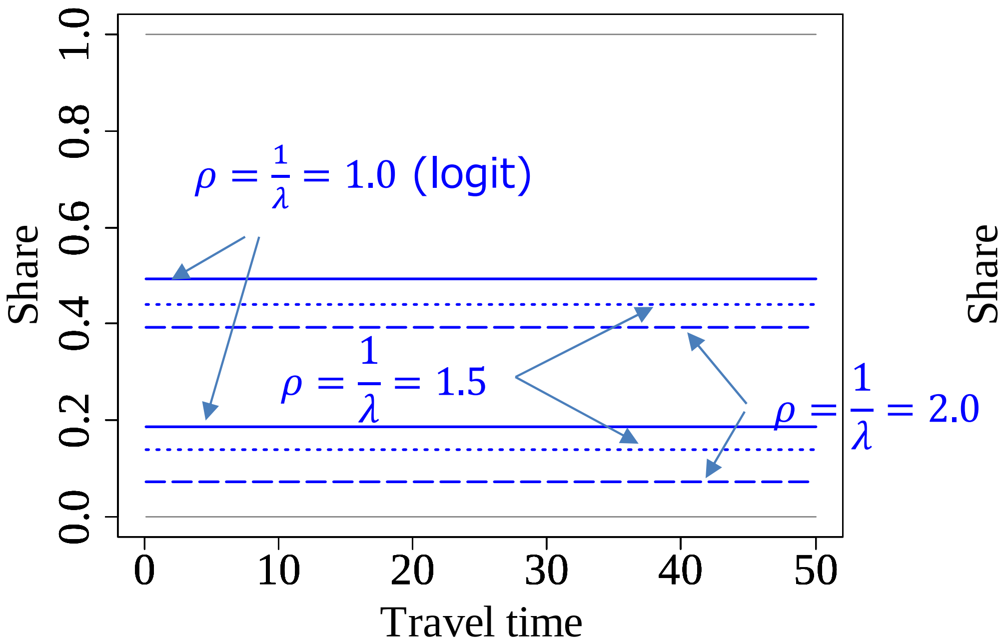
Weibit



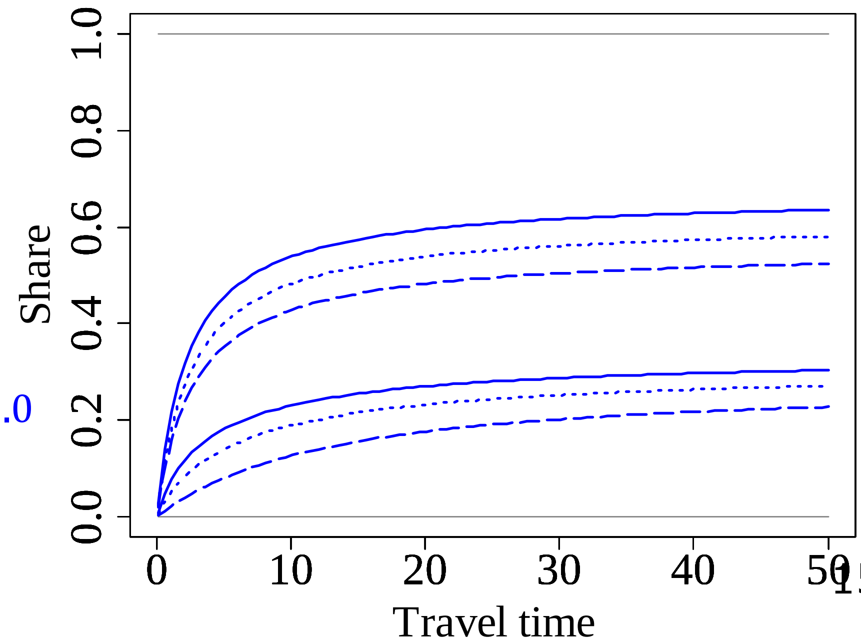
Illustration



Nested logit

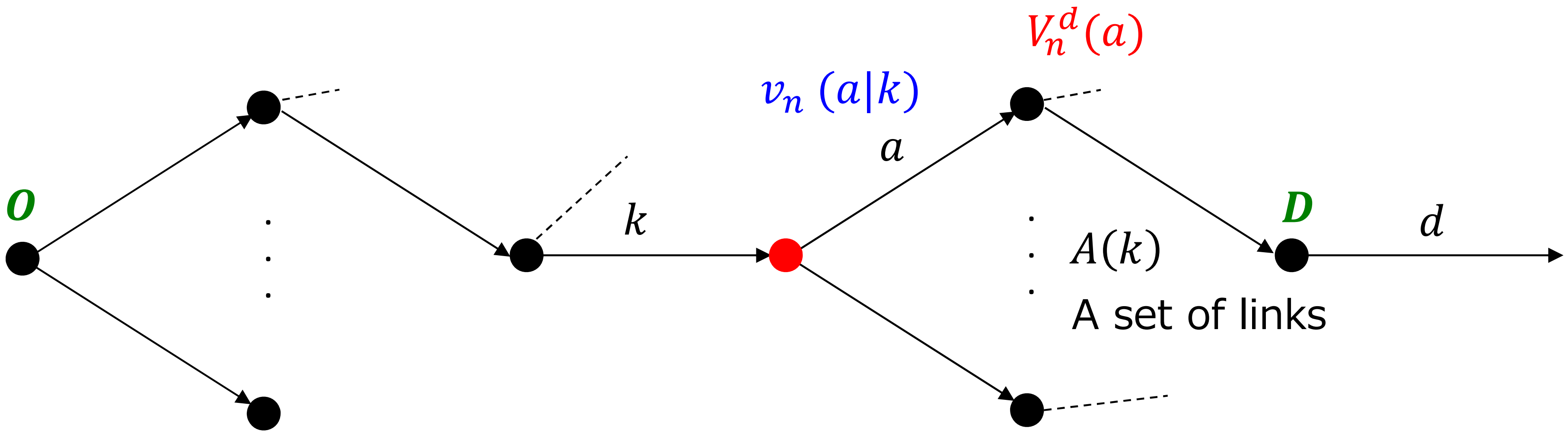


Nested weibit



Recursive logit

The recursive logit model corresponds to a dynamic discrete choice model where the path choice problem is formulated as a sequence of link choices (same as Akamatsu (1996))



$$u(a|k) = v(a|k) + V(a) + \mu\varepsilon(a)$$

where $\quad V(k) = E[\max_{a \in A(k)} \big(v(a|k) + V(a) + \mu\varepsilon(a)\big)]$

Instantaneous cost

i.i.d. error terms (Gumbel)

The expected maximum utility to the destination

Fasgerau et al. (2013)

Recursive logit

$$u(a|k) = v(a|k) + V(a) + \mu\varepsilon(a)$$

where $$V(k) = E[\max_{a\in A(k)}\left(v(a|k) + V(a) + \mu\varepsilon(a)\right)]$$

Link choice Probability:

$$P(a|k) = \frac{e^{\frac{1}{\mu}(v(a|k)+V(a))}}{\sum_{a\prime\in A(k)} e^{\frac{1}{\mu}(v(a\prime|k)+V(a\prime))}}$$

Route choice probability:

$\sigma = \{k_i\}_{i=0}^{I}$

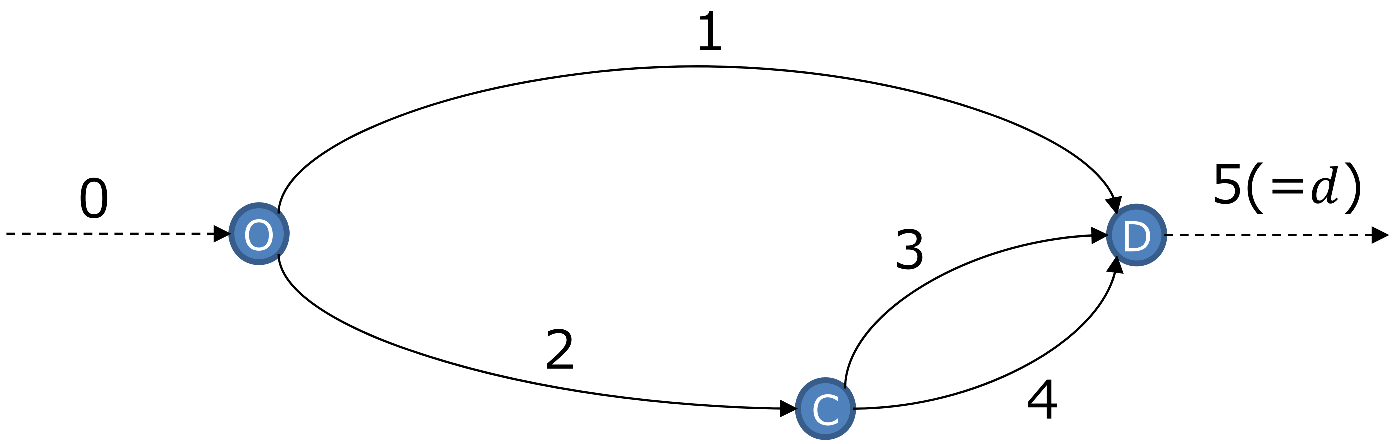
$$P(\sigma) = \prod_{i=0}^{I-1} P(k_{i+1}|k_i) = \prod_{i=0}^{I-1} e^{v(k_{i+1}|k_i)+V(k_{i+1})-V(k_i)}$$
$$= e^{-V(k_0)} \prod_{i=0}^{I-1} e^{v(k_{i+1}|k_i)}$$

Log-likelihood:

$$LL(\beta) = \ln \prod_{n=1}^{N} P(\sigma_n)$$
$$= \frac{1}{\mu} \sum_{n=1}^{N} \left(\sum_{i=0}^{I_n-1} v(k_{i+1}|k_i) - V(k_0) \right)$$

Can be analytically obtained

Illustration



Incidence matrix $\mathbf{L}$

$$\underbrace{k}_{}\overbrace{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}^{a}$$

Vector of the expected maximum utility $\mathbf{Z}$ ($\mu = 1$)

$$\overbrace{\begin{pmatrix} e^{V(0)} = \sum_{a\in A} L_{0a}\cdot e^{v(a|0)+V(a)} \\ e^{V(1)} = \sum_{a\in A} L_{1a}\cdot e^{v(a|1)+V(a)} \\ e^{V(2)} = \sum_{a\in A} L_{2a}\cdot e^{v(a|2)+V(a)} \\ e^{V(3)} = \sum_{a\in A} L_{3a}\cdot e^{v(a|3)+V(a)} \\ e^{V(4)} = \sum_{a\in A} L_{4a}\cdot e^{v(a|4)+V(a)} \\ e^{V(d)} = 1 \end{pmatrix}}^{1} \Bigg\} k$$

Matrix defining instantaneous utility $\mathbf{M}$ ($\mu = 1$)

$$\overbrace{\begin{pmatrix} 0 & e^{v(1|0)} & e^{v(2|0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{0} \\ 0 & 0 & 0 & e^{v(3|2)} & e^{v(4|2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{0} \\ 0 & 0 & 0 & 0 & 0 & e^{0} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}^{a} \Bigg\} k$$

Elements of $\mathbf{z}$ and $\mathbf{M}$:

$$z_k = e^{V(k)} = \begin{cases} \sum_{a\in A} L_{ka}\cdot e^{v(a|k)+V(a)} & \forall k \in A \\ 1 & k = d \end{cases}$$

$$M_{ka} = \begin{cases} L_{ka}\cdot e^{v(a|k)} & a \in A(k) \\ 0 & k = d \end{cases}$$

$$\mathbf{z} = \mathbf{Mz} + \mathbf{b} \rightarrow \mathbf{z} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{b} \leftarrow \mathbf{b}' = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$$

$V(k)$ can be analytically obtained

Generalization of recursive logit

Recursive logit (Fosgerau et al., 2013)

$$u(a|k) = v(a|k) + V(a) + \mu\varepsilon(a)$$

where $$V(k) = E\left[\max_{a\in A(k)}\left(v(a|k) + V(a) + \mu\varepsilon(a)\right)\right]$$

Nested recursive logit (Mai et al., 2015)

$$u(a|k) = v(a|k) + V(a) + \mu_k\varepsilon(a)$$

where $$V(k) = E\left[\max_{a\in A(k)}\left(v(a|k) + V(a) + \mu_k\varepsilon(a)\right)\right]$$

Generalized recursive logit (Mai, 2016)

$$u(a|k) = v(a|k) + V(a) + \mu\varepsilon(a)$$

where $$V(k) = E\left[\max_{a\in A(k)}\left(v(a|k) + V(a) + \varepsilon(a|k) - \frac{\gamma}{\mu_k}\right)\right]$$

Following the MEV distribution (expressed through G function)

Generalization leads to difficulties in model estimation

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